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Variation Diminishing Properties of Bernstein Polynomials on a Tetrahedron

A. BHATT AND A. OJHA*

Department of Mathematics and Computer Science, R. D. University, Jabalpur 482001, India

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Goodman has recently studied certain variation diminishing properties of Bernstein polynomials on triangles. Introducing analogous definitions for the variation of a trivariate function, we study in the present paper corresponding results for the Bernstein polynomials defined on a tetrahedron. We have also extended these results to arbitrary dimension.

1. INTRODUCTION

Goodman [3, 4] has recently studied certain variation diminishing properties of Bernstein polynomials on triangles. For a bivariate suitably smooth function f defined on a triangle T, he introduced the following definitions as analogues of total variations $V(g, [a, b])$ and $V_1(g, [a, b]) = V(g', [a, b])$ of a univariate function g defined on $[a, b]$ and its derivative g':

$$V(f, T) = \int_T |\nabla f| \, dx, \tag{1.1}$$

$$V_1(f, T) = \int_T (|\nabla f_{x_1}|^2 + |\nabla f_{x_2}|^2) \, dx. \tag{1.2}$$

Here $x = (x_1, x_2) \in T$ and ∇f denotes the gradient of f, while f_{x_j} denotes the partial derivative of f with respect to $x_j, j = 1, 2$. For a function f which does not belong to $C^2(T)$ but has discontinuities in the first partial derivatives across a certain line segment ℓ, the variation of f over ℓ was also defined in [3] by

$$V_1(f, \ell) = \int_\ell |\nabla f_1 - \nabla f_2| \, dx, \tag{1.3}$$

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where f_1 and f_2 denote the restrictions of f to either side of ℓ. Then for a function f which is C^2 on T except for having discontinuities in the first derivatives across certain line segments $\ell_1, ..., \ell_m$ in T, its variation over T was defined by

$$V_1(f, T) = V_1(f, T - (\ell_1 \cup \ell_2 \cdots \cup \ell_m)) + \sum_{i=1}^{m} V_1(f, \ell_i). \tag{1.4}$$

We denote the class of all such functions by $D_2(T)$. It has been shown in [3] that for any function f defined on T,

$$V(B_n(f), T) \leqslant (2n/(n+1))\, V(\hat{f}_n, T), \tag{1.5}$$

and

$$V_1(B_n(f), T) \leqslant V_1(\hat{f}_n, T), \tag{1.6}$$

where $B_n(f)$ and $\hat{f}_n$, respectively, denote the nth Bernstein polynomial and nth Bézier net corresponding to f on T.

In a subsequent paper [4], Goodman has considered a generalization of (1.2) by defining

$$V_S(f, T) = \int_T S(f_{x_1x_1}, f_{x_1x_2}, f_{x_2x_2})\, dx, \tag{1.7}$$

where S is a seminorm on $\mathbb{R}^3$ and $f_{x_ix_j}$ denotes the second partial derivative of f with respect to x_i, x_j. For a function having discontinuities in the first derivatives across a line segment ℓ, the variation over ℓ has been defined by

$$V_S(f, \ell) = S(\mu^2, \mu\nu, \nu^2) \int_\ell |\nabla f_1 - \nabla f_2|\, dx, \tag{1.8}$$

where (μ, ν) is a unit vector orthogonal to ℓ (cf. [4, p. 299]). Then given a function $f \in D_2(T)$, $V_s(f, T)$ has been defined as

$$V_S(f, T) = V_S(f, T - (\ell_1 \cup \cdots \cup \ell_m)) + \sum_{i=1}^{m} V_S(f, \ell_i). \tag{1.9}$$

It may be mentioned here that if we consider $S(x) = (x_1^2 + 2x_2^2 + x_3^2)^{1/2}$, then $V_S(f, T)$ reduces to $V_1(f, T)$, while $S(x) = |x_1 + x_3|$ corresponds to $V_1^*(f, T)$ introduced by Chang and Hoschek [1].

Using the foregoing definitions, Goodman has obtained the following result: For any $n \geqslant 1$ and any function f defined on T,

$$V_S(B_n(f), T) \leqslant V_S(\hat{f}_n, T). \tag{1.10}$$

Introducing analogous definitions for d-variate functions ($d \geqslant 3$) defined on a d-simplex, we show in Section 3 that the variation $V_S(B_n(f), T)$ of the Bernstein polynomial over a tetrahedron is bounded by $2n/(n+1)$ times the variation of its Bézier-net. We also give an example to show that the constant $2n/(n+1)$ is the best possible. In Section 4, we have determined a bound for $V_S(B_n(f), T)$ in case when T is a d-simplex, for arbitrary d. Finally, we show in Section 5 that an inequality similar to (1.5) holds for the d-variate case too, except that the constant $2n/(n+1)$ is replaced by $n^d \binom{n+d-1}{n-1}^{-1}$.

We begin with certain definitions, notations, and a result due to Dahmen and Micchelli [2], which is needed in our subsequent discussions.

2. Definitions, Notations, and Some Preliminary Results

Let f be a suitably smooth function on a region $\Omega \subset \mathbb{R}^d$. We introduce the following:

$$V(f, \Omega) = \int_\Omega |\nabla f| \, dx, \tag{2.1}$$

$$V_S(f, \Omega) = \int_\Omega S(\sigma f) \, dx. \tag{2.2}$$

Here ∇f denotes gradient of f, S is a seminorm on $\mathbb{R}^{q(d)}$, $q(d) = d(d+1)/2$, $x = (x_1, ..., x_d) \in \Omega$, and σf is a $q(d)$-tuple given by $\sigma f = (f_{x_i x_j})_{1 \leqslant i \leqslant j \leqslant d}$. For $x \in \Omega$, we shall write $S(x)$ for $S((x_i x_j)_{1 \leqslant i \leqslant j \leqslant d})$. For a function having discontinuities in its first derivatives across a hyperplane P in Ω, we define the analogue of (1.8) by

$$V_S(f, P) = S(a) \int_P |\nabla f_1 - \nabla f_2| \, ds, \tag{2.3}$$

where a is a unit vector orthogonal to the hyperplane P and f_1, f_2 denote the restrictions of f to either side of P. Then for a function f which belongs to C^2 on Ω except having discontinuities in its first derivatives across certain hyperplanes $P_1, ..., P_m$ in Ω, its variation over Ω may be defined as

$$V_S(f, \Omega) = V_S(f, \Omega - (P_1 \cup \cdots \cup P_m)) + \sum_{i=1}^m V_S(f, P_i). \tag{2.4}$$

Let us consider a d-simplex T with vertices $x^i = (x_1^i, ..., x_d^i)$, $i = 1, ..., d+1$. Denoting by $\lambda = (\lambda_1, ..., \lambda_{d+1})$, the barycentric coordinates of a point $x \in T$, we introduce the Bernstein polynomial $B_n(f)$ of f over T as

$$B_n(f)(\lambda) = \sum_{|\alpha| = n} \binom{n}{\alpha} f(\alpha/n)\, \lambda^\alpha, \tag{2.5}$$

where $\alpha = (\alpha_1, ..., \alpha_{d+1}) \in \mathbb{Z}_+^{d+1}$ with $|\alpha| = \alpha_1 + \cdots + \alpha_{d+1}$, $\binom{n}{\alpha} = (n!/\prod_{j=1}^{d+1} (\alpha_j!))$, while $\lambda^\alpha = \prod_{j=1}^{d+1} \lambda_j^{\alpha_j}$.

We now turn to introduce the d-dimensional analogue of $\hat{f}_n$. For this, we first observe that there is no unique way of defining regular triangulations in the d-dimensional case. In view of this, we consider the following canonical way to construct triangulation for an arbitrary d-simplex ($d \geqslant 2$) (cf. [2, p. 273]). We let $\mathscr{P}_d$ denote the group of all permutations of $\{1, ..., d\}$, and for $\pi \in \mathscr{P}_d$, we define the simplex

$$\begin{aligned} \delta_\pi &= \{u \in [0, 1]^d : u_{\pi(1)} \geqslant \cdots \geqslant u_{\pi(d)}\} \\ &= [v^0, ..., v^d], \end{aligned}$$

where $v^0 = 0$, $v^j = v^{j-1} + e^{\pi(j)}$, $j = 1, ..., d$. We note that all the simplexes are congruent and the simplex δ_i corresponding to the identity $i \in \mathscr{P}_d$ is given by

$$\begin{aligned} \delta_i &= \{u \in [0, 1]^d : u_1 \geqslant \cdots \geqslant u_d\} \\ &= [0, e^1, e^1 + e^2, ..., e^1 + e^2 + \cdots + e^d]. \end{aligned}$$

We next see that for any positive integer k,

$$C_{d,k} = \{\delta/k : \delta \in K_d, \delta \subset k\delta_i\},$$

is a triangulation of δ_i (cf. [2, p. 274]), where

$$K_d = \{\delta_\pi + \alpha : \alpha \in \mathbb{Z}^d, \pi \in \mathscr{P}_d\}.$$

Thus for any arbitrary simplex $\sigma \in \mathbb{R}^d$, there exists an affine map $A: \delta_i \to \sigma$ such that the set $C_{d,k}(A) = \{A(\delta) : \delta \in C_{d,k}\}$ is a triangulation of σ. For the simplex T, we define the mapping A_T by requiring that $A_T(0) = x^1$, $A_T(e^1) = x^2$, $A_T(e^1 + e^2) = x^3$, $A_T(e^1 + e^2 + \cdots + e^d) = x^{d+1}$. We can now define $\hat{f}_n$ as the piecewise linear interpolant of f with respect to the triangulation $C_{d,n}(A_T)$, interpolating f at the points whose barycentric coordinates are $\{\alpha/n : |\alpha| = n\}$. The following lemma due to Dahmen and Micchelli [2, p. 274] is useful in determining $V_S(\hat{f}_n, T)$.

LEMMA 2.1. *For any two simplexes* $\delta = [u^1, u^2, ..., u^{d+1}]$ *and* $\tilde{\delta} = [\tilde{u}^1, u^2, ..., u^{d+1}]$ *in* $C_{d,k}(A_T)$, *there exist vertices* $u^p, u^q \in \delta \cap \tilde{\delta}$ *such that* $u^1, \tilde{u}^1, u^p, u^q$ *span a planar parallelogram.*

3. BOUND FOR $V_S(B_n(f), T)$ WHEN $d = 3$

We are now in a position to state the following.

THEOREM 3.1. *For any* $n \geqslant 1$,

$$V_S(B_n(f), T) \leqslant (2n/(n+1))\, V_S(\hat{f}_n, T). \tag{3.1}$$

Before we give the proof of the foregoing theorem, we need to introduce some additional notations which will be required in this section.

We define the following:

$$\Delta_{ij}(l\ m\ n) = \begin{vmatrix} 1 & 1 & 1 \\ x_i^l & x_i^m & x_i^n \\ x_j^l & x_j^m & x_j^n \end{vmatrix},$$

$l, m, n \in \{1, 2, 3, 4\}$ and $i, j \in \{1, 2, 3\}$. It may be seen that

$$\nabla\lambda_1 = -(\Delta_{23}(234), \Delta_{31}(234), \Delta_{12}(234))/6\Delta,$$

where Δ denotes the volume of T. $\nabla\lambda_i$ $(i = 2, 3, 4)$ have similar expressions. For convenience, we shall write $\nabla\lambda_i = (\gamma_1^i, \gamma_2^i, \gamma_3^i)/6\Delta = \gamma^i/6\Delta$, say. We will also use the notation Δ_i for the area of the face $(x^{i+1}, x^{i+2}, x^{i+3})$, $i \in \mathbb{Z}_4$ (additive group of integers modulo 4). It can be seen that $\Delta_i = (1/2)\,|\gamma^i|$. We also set

$$\gamma^5 = \gamma^1 + \gamma^2, \qquad \gamma^6 = \gamma^2 + \gamma^3.$$

For convenience, we shall write $f(\alpha)$ for $f(\alpha/n)$ in our subsequent discussions. We set $E_i f(\alpha) = f(\alpha + e^i)$, $i = 1, 2, 3, 4$, where $\{e^i\}$ is the standard canonical basis for $\mathbb{R}^4$. Using shift operators E_i, we introduce the following:

$$\begin{aligned} D_1 &= (E_1 - E_2)(E_1 - E_4), & D_2 &= (E_2 - E_1)(E_2 - E_3), \\ D_3 &= (E_3 - E_2)(E_3 - E_4), & D_4 &= (E_4 - E_1)(E_4 - E_3), \\ D_5 &= (E_1 - E_4)(E_2 - E_3), & D_6 &= (E_1 - E_2)(E_4 - E_3). \end{aligned} \tag{3.2}$$

Proof. We first determine $V_S(\hat{f}_n, T)$. For this, we consider the restriction of $\hat{f}_n$ over any two subsimplexes $[u^1, u^2, u^3, u^4]$ and $[\tilde{u}^1, u^2, u^3, u^4]$ in T. Using Lemma 2.1, we see after some simplifications that the magnitude of the change in gradient across the common face $[u^2, u^3, u^4]$ is given by

$$n^3\, |f(u^1)+f(\tilde{u}^1)-f(u^p)-f(u^q)|\; \Delta(u^2, u^3, u^4)/3\Delta, \tag{3.3}$$

where u^p, u^q are as in Lemma 2.1 and $\Delta(u^2, u^3, u^4)$ is the area of the common face.

We note that the common faces between any two simplexes in $C_{3,n}(A_T)$ lie on planes having six different slopes. Thus the magnitude of change in gradient of $\hat{f}_n$ across a common face lying on a plane $n\lambda_j=\beta_j$, $1\leqslant\beta_j\leqslant n-1$ is given by

$$n\,|D_j f(\alpha)|\; \Delta_j/3\Delta, \qquad |\alpha|=n-2. \tag{3.4}$$

We next observe that some of the faces lie on planes which are not of the type $n\lambda_j=\beta_j$. These planes are

$$(\lambda_1+\lambda_2)(\alpha_3+\alpha_4)-(\lambda_3+\lambda_4)(\alpha_1+\alpha_2)=0, \tag{3.5}$$

$$(\lambda_2+\lambda_3)(\alpha_1+\alpha_4)-(\lambda_1+\lambda_4)(\alpha_2+\alpha_3)=0, \tag{3.6}$$

for $|\alpha|=n-2$. For a face lying on a plane of the type (3.5), the magnitude of the change in gradient of $\hat{f}_n$ is given by

$$n\,|D_5 f(\alpha)|\; |\gamma^5|/6\Delta, \tag{3.7}$$

while that across a face of the type (3.6) is given by

$$n\,|D_6 f(\alpha)|\; |\gamma^6|/6\Delta. \tag{3.8}$$

Since γ^i is orthogonal to the face $[x^{i+1}, x^{i+2}, x^{i+3}]$, $i\in\mathbb{Z}_4$, while $\gamma^5=\gamma^1+\gamma^2$ and $\gamma^6=\gamma^2+\gamma^3$ are orthogonal to the planes of the type (3.5) and (3.6), respectively, we have

$$V_S(\hat{f}_n, T)\geqslant(1/12n\Delta)\sum_{|\alpha|=n-2}\;\sum_{k=1}^{6} S(\gamma^k)\,|D_k f(\alpha)|. \tag{3.9}$$

We next consider $B_n(f)_{x_i x_j}$, $i, j=1, 2, 3$. A direct computation gives

$$B_n(f)_{x_i x_j}=(n(n-1)/36\Delta^2)\sum_{|\alpha|=n-2}\binom{n-2}{\alpha}\lambda^\alpha(\gamma_i\cdot E)(\gamma_j\cdot E)f(\alpha),$$

where $\gamma_i\cdot E=\gamma_i^1E_1+\gamma_i^2E_2+\gamma_i^3E_3+\gamma_i^4E_4$. Using the fact that $\sum_{k=1}^4\gamma_j^k=0$, we may express $B_n(f)_{x_i x_j}$ as

$$B_n(f)_{x_i x_j}=(n(n-1)/36\Delta^2)\sum_{|\alpha|=n-2}\binom{n-2}{\alpha}\lambda^\alpha\sum_{k=1}^{6}\gamma_i^k\gamma_j^k D_k f(\alpha). \tag{3.10}$$

Thus

$$S(\sigma B_n(f)) \leqslant (n(n-1)/36\Delta^2) \sum_{|\alpha|=n-2} \binom{n-2}{\alpha} \lambda^\alpha \sum_{k=1}^{6} S(\gamma^k)\, |D_k f(\alpha)|, \quad (3.11)$$

which gives

$$V_S(B_n(f), T) \leqslant (1/6(n+1)\,\Delta) \sum_{|\alpha|=n-2} \sum_{k=1}^{6} S(\gamma^k)\, |D_k f(\alpha)|. \quad (3.12)$$

The last step follows since for $|\alpha| = n-2$,

$$\int_T \binom{n-2}{\alpha} \lambda^\alpha\, dx = 6\Delta/(n-1)\, n(n+1). \quad (3.13)$$

Comparing (3.9) and (3.12), we obtain (3.1).

We now give a simple example to show that $V_S(B_n(f), T)$ can not be bounded by $V_S(\hat{f}_n, T)$, in general. For this, we consider a function f such that $f(x^1) = 1$ and $f(x^i) = 0$, $i = 2, 3, 4$. Also $f((x^i + x^j)/2) = 0$. Then $V_S(\hat{f}_2, T) = S(\gamma^1)/24\Delta$ while $V_S(B_2(f), T) = S(\gamma^1)/18\Delta$.

4. Bound for $V_S(B_n(f), T)$: Arbitrary d

The following notations will be needed in the present and the next sections.

For any set $K \subset \mathbb{R}^d$, $\operatorname{vol}_k K$ denotes the k-dimensional volume of K $(k \leqslant d)$. As in the previous section, we set $E_i f(\alpha) = f(\alpha + e^i)$, $i = 1, ..., d+1$, where $\{e^i\}$ is the standard canonical basis for $\mathbb{R}^{d+1}$. We write $\nabla\lambda_i = \gamma^i/(d!\, \Delta)$. We also write

$$D_{i,j} = -(E_i - E_{i+1})(E_j - E_{j+1}), \qquad 1 \leqslant i < j \leqslant d+1, \quad E_{d+2} = E_1.$$

We note that $|\gamma^i| = (d-1)!\, \Delta_i$, where $\Delta_i = \operatorname{vol}_{d-1} T_i$ and T_i is the face of the simplex which does not contain the vertex x^i. If $[u^1, u^2, ..., u^{d+1}]$ and $[\tilde{u}^1, u^2, ..., u^{d+1}]$ are two subsimplexes in $C_{d,n}(A_T)$, then the absolute value of the change in gradient across the common face $[u^2, ..., u^{d+1}]$ is given by

$$(n^d/d\Delta)\, |f(u^1) + f(\tilde{u}^1) - f(u^p) - f(u^q)|\, \operatorname{vol}_{d-1}[u^2, ..., u^{d+1}]$$

where u^p, u^q are as in Lemma 2.1 and $\Delta = \operatorname{vol}_d T$.

After some calculations, one can see that the variation $V_S(\hat{f}_n, T)$ satisfies

$$V_S(\hat{f}_n, T) \geqslant (1/d!\,(d-1)!\,\Delta n^{d-2}) \sum_{|\alpha|=n-2} \sum_{1 \leqslant k < m \leqslant d+1} |D_{k,m} f(\alpha)|\, S(\gamma^{k,m}), \tag{4.1}$$

where $\gamma^{i,j} = \sum_{q=i+1}^{j} \gamma^q$. We also have

$$B_n(f)_{x_i x_j} = (n((n-1)/(d!\,\Delta)^2) \sum_{|\alpha|=n-2} \binom{n-2}{\alpha} \lambda^\alpha (\gamma_i \cdot E)(\gamma_j \cdot E) f(\alpha),$$

where $\gamma_i \cdot E = \gamma_i^1 E_1 + \cdots + \gamma_i^{d+1} E_{d+1}$. Using the fact that $\sum_{k=1}^{d+1} \gamma_j^k = 0$, we may express $B_n(f)_{x_i x_j}$ as

$$B_n(f)_{x_i x_j} = (n(n-1)/(d!\,\Delta)^2) \sum_{|\alpha|=n-2} \binom{n-2}{\alpha} \times \lambda^\alpha \sum_{1 \leqslant k < m \leqslant d+1} D_{k,m} f(\alpha)\, \gamma_i^{k,m} \gamma_j^{k,m}.$$

This gives

$$S(\sigma B_n(f)) \leqslant (n(n-1)/(d!\,\Delta)^2) \sum_{|\alpha|=n-2} \binom{n-2}{\alpha} \lambda^\alpha \times \sum_{1 \leqslant k < m \leqslant d+1} |D_{k,m} f(\alpha)|\, S(\gamma^{k,m}).$$

We thus have

$$V_S(B_n(f), T) \leqslant \left(1/d!\,\Delta \prod_{j=1}^{d-2} (n+j) \right) \times \sum_{|\alpha|=n-2} \sum_{1 \leqslant k < m \leqslant d+1} |D_{k,m} f(\alpha)|\, S(\gamma^{k,m}). \tag{4.2}$$

This follows, since for $|\alpha| = n$,

$$\int_T \binom{n}{\alpha} \lambda^\alpha \, dx = d!\,\Delta \Big/ \prod_{j=1}^{d} (n+j). \tag{4.3}$$

Combining (4.1) and (4.2), we obtain

THEOREM 4.1. *For any function f defined on a d-simplex,*

$$V_S(B_n(f), T) \leqslant C(n, d)\, V_S(\hat{f}_n, T),$$

where $C(n, d) = n^{d-1} \binom{n+d-2}{n-1}^{-1}$.

We now proceed to determine a bound for $V(B_n(f), T)$. Denoting by U_α the subsimplex of T with vertices at $(\alpha+e^1)/n$, $(\alpha+e^2)/n$, ..., $(\alpha+e^{d+1})/n$, we set

$$U_n(T) = \bigcup \{U_\alpha : |\alpha| = n-1\}.$$

5. Bound for $V(B_n(f), T)$

We first determine $V(B_n(f), T)$. We have

$$B_n(f)_{x_j} = (n/d!\,\Delta) \sum_{|\alpha|=n-1} \binom{n-1}{\alpha} \lambda^\alpha (\gamma_j \cdot E) f(\alpha), \tag{5.1}$$

for $j = 1, ..., d$. An application of triangle inequality gives

$$|\nabla B_n(f)| \leqslant (n/d!\,\Delta) \sum_{|\alpha|=n-1} \binom{n-1}{\alpha} \lambda^\alpha \left(\sum_{i=1}^{d} ((\gamma_i \cdot E) f(\alpha))^2 \right)^{1/2}.$$

We next notice that $\nabla\lambda_i = n\gamma^i/d!\,\Delta$, where $\{\lambda_i\}$ are the barycentric coordinates of a point with respect to U_α. Using this and (4.3), we obtain

$$\begin{aligned}
\int_T |\nabla B_n(f)|\, dx &\leqslant \left(1 \Big/ \prod_{j=1}^{d-1} (n+j)\right) \sum_{|\alpha|=n-1} \left(\sum_{i=1}^{d} ((\gamma_i \cdot E) f(\alpha))^2 \right)^{1/2} \\
&= \left(d!\, n^{d-1} \Big/ \prod_{j=1}^{d-1} (n+j)\right) \sum_{|\alpha|=n} \int_{U_\alpha} |\nabla \hat{f}_n|\, dx \\
&= \left(d!\, n^{d-1} \Big/ \prod_{j=1}^{d-1} (n+j)\right) V(\hat{f}_n, U_n(T)).
\end{aligned}$$

This proves the following.

Theorem 5.1. *For any $n \geqslant 1$, and any function f defined on T,*

$$V(B_n(f), T) \leqslant C(n, d+1)\, V(\hat{f}_n, T).$$

It is easy to see that the foregoing theorem remains valid if we replace $V(B_n(f), T)$ and $V(\hat{f}_n, T)$ by $V_{\tilde{S}}(B_n(f), T)$ and $V_{\tilde{S}}(\hat{f}_n, T)$, respectively, where $\tilde{S}$ is a seminorm defined on $\mathbb{R}^d$ and for any suitably smooth function f,

$$V_{\tilde{S}}(f, T) = \int_T \tilde{S}(\nabla f)\, dx.$$

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